3.4 Linear equation systems

Definition: A system,

Linear equation system with m equations and n variables.

If all b_i for i = 1, 2, ..., m are equal to 0,then the linear equation system is called **homogeneous**, otherwise it is called **inhomogeneous**.

Vector presentation:

 $\underline{\mathbf{a}}_1 \mathbf{x}_1 + \underline{\mathbf{a}}_2 \mathbf{x}_2 + \dots + \underline{\mathbf{a}}_n \mathbf{x}_n = \underline{\mathbf{b}}$

with
$$\underline{\mathbf{a}}_{i} = \begin{pmatrix} \mathbf{a}_{1i} \\ \mathbf{a}_{2i} \\ \vdots \\ \mathbf{a}_{mi} \end{pmatrix}$$
, für $i = 1, 2, ..., n$ und $\underline{\mathbf{b}} = \begin{pmatrix} \mathbf{b}_{1} \\ \mathbf{b}_{2} \\ \vdots \\ \mathbf{b}_{m} \end{pmatrix}$.

Matrix presentation:

 $\underline{\mathbf{A}} \underline{\mathbf{x}} = \underline{\mathbf{b}}$

with
$$\underline{\mathbf{A}} = \begin{pmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} & \cdots & \mathbf{a}_{1n} \\ \mathbf{a}_{21} & \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & & & \vdots \\ \mathbf{a}_{m1} & \mathbf{a}_{m2} & \cdots & \mathbf{a}_{mn} \end{pmatrix}$$

The term of a solution of a linear equation system:

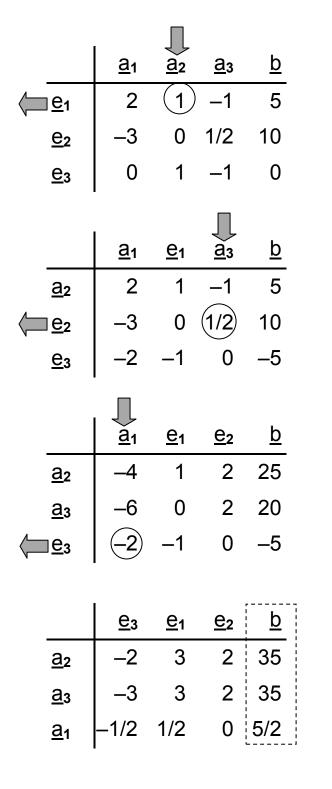
Definition: a vector $\mathbf{x} = \hat{\mathbf{x}}$ of fixed values, which satisfies the condition $\underline{A} \hat{\underline{x}} = \underline{b}$ (which transfers it in an identity), is called a **solution** of the linear equation system $\underline{A} \underline{x} = \underline{b}$.

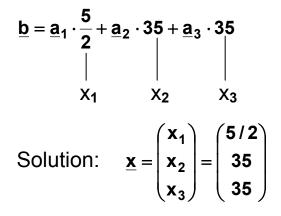
Example:

$$2x_1 + x_2 - x_3 = 5$$

-3x_1 + $\frac{1}{2}x_3 = 10$
x_2 - x_3 = 0

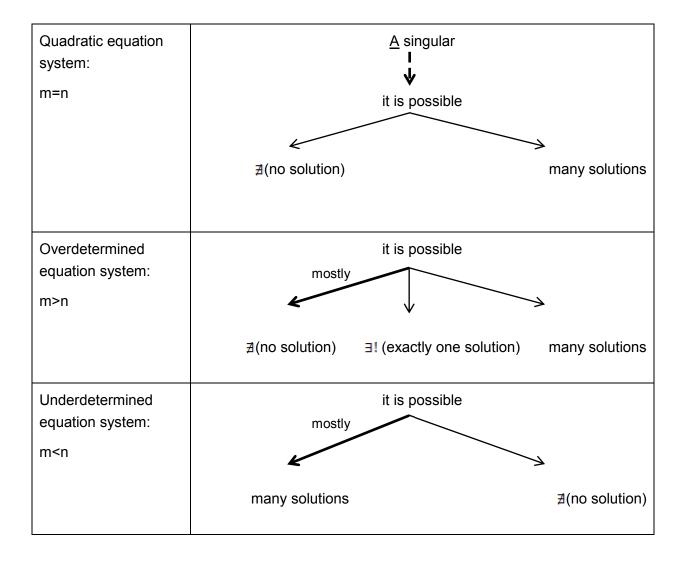
We solve the linear equation system by using elementary transformations of a basis:





(Proof!)

We had three equations and three variables in the example above. There was exactly one solution, which is not always the case.



Two questions:

- a) Solvability: Does the linear equation system have a solution?
- b) Uniqueness: How many solutions does the linear equation system have?

Theorem: A linear equation system $\underline{A} \times = \underline{b}$ is solvable, if and only if $\rho(\underline{A}) = \rho(\underline{A}, \underline{b})$

<u>A,b</u> is called extended coefficient matrix.

In the example above it is $\rho(\underline{A}) = 3 = \rho(\underline{A},\underline{b})$.

Theorem: Given the linear equation systems $\underline{A} \underline{x} = \underline{b}$ with n variables ($\underline{x} \in \mathbb{R}^n$). The linear equation system has exactly one solution if and only if $\rho(\underline{A}) = n$

If, in contrast, it holds $\rho(\underline{A}) = \rho(\underline{A},\underline{b}) = r < n$, then f = n - r variables are free to choose, viz. we have an infinite number of solutions.

f is the degree of freedom of the linear equation system

Examples:

1) task above: $\rho(\underline{A}) = 3 = n$

2) $3x_1 - x_2 + 2x_3 + x_4 = 5$

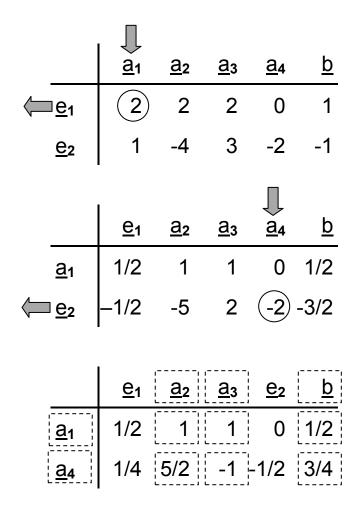
The linear equation system is composed of one equation with four variables.

 $\rho(\underline{A}) = \rho(\underline{A},\underline{b}) = 1$, n = 4, f = 3; $x_4 = 5 - 3x_1 + x_2 - 2x_3$, $x_1, x_2, x_3 \in \mathbb{R}$ arbitrary

3) $2x_1 + 2x_2 + 2x_3 = 1$ $x_1 - 4x_2 + 3x_3 - 2x_4 = -1$

The linear equation system is composed of two equations with four variables.

Solution of the linear equation system using elementary transformations of a basis:



General solution:

We get special solutions by fixing the arbitrary variables:

a)
$$x_2 = 1$$

 $x_3 = 1$
b) $x_2 = 0$
 $x_4 = -\frac{3}{4}$
 $x_4 = -\frac{3}{4}$
 $x_4 = -\frac{3}{4}$
 $x_1 = -\frac{1}{2}$
 $x_1 = -\frac{1}{2}$
 $x_4 = \frac{7}{4}$
 $x_4 = \frac{7}{4}$
 $x_1 = -\frac{3/2}{1}$
 $x_1 = -\frac{1}{2}$
 $x_2 = 0$
 $x_2 = 0$
 $x_2 = 0$
 $x_3 = 1$
 $x_4 = \frac{7}{4}$